On Krein-von Neumann and Friedrichs extensions

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To the Memory of László György Pál

Abstract. We give domain and range characterization of the smallest (Krein-von Neumann) positive self-adjoint extension of Hilbert space operators. There are given new proofs of these facts for the largest (Friedrichs) extension as well. Boundedly or compactly invertible extremal extensions are also characterized as a result of an application of our treatment.

1 Introduction

Given a positive linear operator T on the complex Hilbert space $\mathcal{H}$ with domain $\operatorname{dom} T$ not necessarily dense in the space $\mathcal{H}$, we require

$$(Tf, f) \geq 0 \text{ for each } f \text{ in } \operatorname{dom} T.$$

Further assumptions (for example closedness of T or $\operatorname{ran} T$) are not posed. But, in any case, the range space will always be given an inner product

$$\langle Tf, Tg \rangle \doteq (Tf, g) \ \text{ for all } f, g \text{ from } \operatorname{dom} T.$$

The corresponding completion Hilbert space will be denoted by $\mathcal{H}_T$. The natural identification operator of $\operatorname{ran} T$, regarded as $J : \mathcal{H}_T \to \mathcal{H}$, determines its adjoint $J^* : \mathcal{H} \to \mathcal{H}_T$ with domain

$$\mathcal{D}_* [T] \doteq \left\{ g \in \mathcal{H} : \sup \left\{ |(Tf, g)|^2 : f \in \operatorname{dom} T, (Tf, f) \leq 1 \right\} < \infty \right\}.$$

If T admits a positive self-adjoint extension (see [2] for closed T, [14]) then $\mathcal{D}_* [T]$ is necessarily dense in the space $\mathcal{H}$ and $J^{**}J^*$ is the smallest (Krein-von Neumann) extension of T such that $\operatorname{dom} (J^{**}J^*)^{1/2} = \mathcal{D}_* [T]$ and $\operatorname{ran} (J^{**}J^*)^{1/2} = \mathcal{R} [T]$ (see [2],[10]) where

$$\begin{aligned} \mathcal{R} [T] \ &\doteq \ \{ g \in \mathcal{H} : \exists \{f_n\} \subset \operatorname{dom} T, (g - Tf_n, g - Tf_n) \to 0, \\ &\quad (Tf_m - Tf_n, f_m - f_n) \to 0 \}. \end{aligned}$$

The largest (Friedrichs) positive self-adjoint extension of T exists if and only if T is densely defined. The problem was initiated by von Neumann [16], solved by Stone [15], Freudenthal [6] and Friedrichs [7]. Krein [9] published a theory of these extensions in the densely defined case based on the Cayley transform reducing the problem to the contractive self-adjoint one (see also [12]). If we assume that $\operatorname{dom} T$ is dense in $\mathcal{H}$ then $Q \doteq J^* |_{\operatorname{dom} T}$ as an $\mathcal{H} \to \mathcal{H}_T$ operator provides the largest (Friedrichs) extension of T (see [11]), Q^*Q^{**}, such that $\operatorname{dom} (Q^*Q^{**})^{1/2} = \mathcal{D} [T]$, $\operatorname{ran}(Q^*Q^{**})^{1/2} = \mathcal{R}_* [T]$ where

$$\begin{aligned} \mathcal{D} [T] \ &\doteq \ \{ f \in \mathcal{H} : \exists \{f_n\} \subset \operatorname{dom} T, (f - f_n, f - f_n) \to 0, \\ &\quad (Tf_m - Tf_n, f_m - f_n) \to 0 \}, \\ \mathcal{R}_* [T] \ &\doteq \ \left\{ g \in \mathcal{H} : \sup \left\{ |(f, g)|^2 : f \in \operatorname{dom} T, (Tf, f) \leq 1 \right\} < \infty \right\}. \end{aligned}$$

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Here, if T is positive and self-adjoint then clearly

$$\begin{aligned} \operatorname{dom} T^{1/2} &= \mathcal{D}_* [T] = \mathcal{D} [T] , \\ \operatorname{ran} T^{1/2} &= \mathcal{R}_* [T] = \mathcal{R} [T] . \end{aligned}$$

The above-mentioned factorization of the extensions initiated in [14], [11] and [10], plays an important role. Applying this we give in the first section domain and range characterizations of the Krein-von Neumann extension T_N of T as well as existence and invertibility conditions for T_N. The latter one is the Friedrichs extendibility of T^{-1} so that T_N^{-1} equals $\left(T^{-1}\right)_F$ in accordance with the Ando-Nishio identity $T_F^{-1} = \left(T^{-1}\right)_N$ in [2], where T is assumed to be densely defined, while T_N is invertible if and only if $\operatorname{dom} T^{-1} = \operatorname{ran} T$ is dense. Theorem 4 extends Corollary 5 of [2] in the sense that Friedrichs theorem can not be applied when T is not densely defined. The second section of the paper deals with the corresponding results for the Friedrichs extension, providing new short and concise proofs using the factorization argument we followed. We will use a further notation:

$$\mathcal{R}^* [T] \doteq \left\{ g \in \mathcal{H} : \sup \left\{ |(f,g)|^2 : f \in \operatorname{dom} T, (Tf,Tf) \leq 1 \right\} < \infty \right\} .$$

Note that $\mathcal{R}^* [T] = \operatorname{ran} T^*$ when T is densely defined, see [13].

The last two sections deal with boundedly respectively compactly invertible Krein-von Neumann and Friedrichs extensions. These are the classical situations appearing in most cases in the literature (see [1], [2], [4], [8] and [9]).

We also mention that other extremal extensions (see [3]) and form sum extensions (see [5]) can be obtained using the main tool of our paper. Otherwise, the argument we use mainly depends on classical results of von Neumann on closed densely defined operators, see [12].

2 The Krein-von Neumann extension

First we give necessary and sufficient conditions for the existence of the Krein-von Neuamnn extension extending the results in [2] and [14].

Applying the corresponding factorization in the construction of this extension we give a complete domain and and range characterization of it.

Finally we characterize Hilbert space operators having invertible Krein-von Neumann extension.

Theorem 1 *Let T be a positive linear operator on the complex Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*
(i) T admits the Krein-von Neumann extension T_N.
(ii) T admits a positive self-adjoint extension.
(iii) T is positively closable in the sense that if $\{f_n\} \subset \operatorname{dom} T$ is a sequence satisfying $(Tf_n, f_n) \to 0$ and $(Tf_m - Tf_n, Tf_m - Tf_n) \to 0$ then $(Tf_n, Tf_n) \to 0$.
(iv) $\mathcal{D}_* [T]$ is dense in $\mathcal{H}$.
If this is the case then

$$\left(T_N^{1/2} g, T_N^{1/2} g \right) = \sup \left\{ |(Tf,g)|^2 : f \in \operatorname{dom} T, (Tf,f) \leq 1 \right\} \left(g \in \operatorname{dom} T_N^{1/2} \right) .$$

Proof. Implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear. If S is a positive self-adjoint extension of T then the first part of (iii) implies $\left(S^{1/2} f_n, S^{1/2} f_n\right) \to 0, \left(S\left(f_m - f_n\right), S\left(f_m - f_n\right)\right) \to 0,$ and using the closedness of $S^{1/2}$ we get $(Sf_n, Sf_n) \to 0$. Assuming (iii) we have that T is really positive in the sense that $f \in \operatorname{dom} T$, $(Tf,f) = 0$ implies $Tf = 0$, see $f_n = f$ $\forall n$. Consequently $(\operatorname{ran} T, \langle \cdot, \cdot \rangle)$ is a pre-Hilbert space. Morover, by assumption, $J : \operatorname{ran} T \to \mathcal{H}$ is a closable operator. According to von Neumann's classical result [12], dom $J^* = \mathcal{D}_* [T]$ (see Introduction) is dense in $\mathcal{H}$. Finally, we show that (iv) implies (i). Assuming (iv) we have again that $f \in \operatorname{dom} T$, $(Tf,f) = 0$ implies $Tf = 0$, because $(Tf,g) = 0$ for each g in $\mathcal{D}_* [T]$ which is a dense set in $\mathcal{H}$. Now the above operator J has a densely defined adjoint J^* therefore, thanks to another classical result of von Neumann [12], $J^{**}J^*$ is a positive self-adjoint operator on $\mathcal{H}$. We are going to show that $J^{**}J^*$ is identical to the Krein-von Neumann extension T_N of T. For each $f, g \in \operatorname{dom} T$ we have that $f \in \mathcal{D}_* [T]$ and

$$\langle Tg, J^* f \rangle = (J (Tg), f) = (Tg, f) = \langle Tg, Tf \rangle ,$$

so $J^* f = Tf \in \mathcal{H}_T$ and

$$J^{**}J^* f = J^{**}(Tf) = J (Tf) = Tf \quad (f \in \operatorname{dom} T).$$

Here we find that

$$\begin{aligned} \operatorname{dom} (J^{**}J^*)^{1/2} &= \operatorname{dom} J^* = \mathcal{D}_* [T], \\ \operatorname{ran} (J^{**}J^*)^{1/2} &= \operatorname{ran} J^{**} = \operatorname{ran} \overline{J} = \mathcal{R} [T]. \end{aligned}$$

We show that $J^{**}J^*$ is the smallest among all the positive self-adjoint operator extensions of T and therefore equals to T_N. For if S is such an extension then replacing T by S above we have

$$\operatorname{dom} S^{1/2} = \mathcal{D}_* [S] = \left\{ g \in \mathcal{H} : \sup \left\{ |(Sf, g)|^2 : f \in \operatorname{dom} S, (Sf, f) \le 1 \right\} < \infty \right\}$$

and clearly $\mathcal{D}_* [S] \subset \mathcal{D}_* [T]$, so $\operatorname{dom} S^{1/2} \subset \operatorname{dom} (J^{**}J^*)^{1/2}$. We also determine the metric behaviour of these square root operators. Let $f \in \operatorname{dom} (J^{**}J^*)^{1/2}$. Then

$$\begin{aligned} \left((J^{**}J^*)^{1/2} f, (J^{**}J^*)^{1/2} f \right) &= \langle J^* f, J^* f \rangle \\ &= \sup \left\{ |\langle Tg, J^* f \rangle|^2 : g \in \operatorname{dom} T, (Tg, g) \le 1 \right\} \\ &= \sup \left\{ |(J (Tg), f)|^2 : g \in \operatorname{dom} T, (Tg, g) \le 1 \right\} \\ &= \sup \left\{ |(Tg, f)|^2 : g \in \operatorname{dom} T, (Tg, g) \le 1 \right\}. \end{aligned}$$

Replacing T by S we see that

$$\left(S^{1/2} f, S^{1/2} f \right) = \sup \left\{ |(Sg, f)|^2 : g \in \operatorname{dom} S, (Sg, g) \le 1 \right\} \quad (f \in \operatorname{dom} S)$$

and obviously

$$\left((J^{**}J^*)^{1/2} f, (J^{**}J^*)^{1/2} f \right) \le \left(S^{1/2} f, S^{1/2} f \right) \quad \left(f \in \operatorname{dom} S^{1/2} \right).$$

Everywhere is essential that S extends T. Thus the theorem follows.

□

Theorem 2 *Let T be an operator having a positive self-adjoint extension and thus having the Krein-von Neumann extension T_N. Then*
(v) $\ker T_N = (\operatorname{ran} T)^{\perp} \subset \mathcal{D}_* [T]$.
(vi) $\operatorname{dom} T_N = \{ g \in \mathcal{D}_* [T] : \forall f \in \operatorname{dom} T, (Tf, g) = (f, h)$ *for some* $h \in \mathcal{R} [T] \}$.
(vii) $\operatorname{ran} T_N = \mathcal{R} [T] \cap \mathcal{R}^* [T]$.
Moreover, for every $h \in \operatorname{ran} T_N$ there exists a unique $g \in \overline{\operatorname{ran} T} \cap \mathcal{D}_* [T]$ such that $(Tf, g) = (f, h)$ for all f in $\operatorname{dom} T$ and $T_N g = h$.

Proof. We have seen in the proof of the previous theorem that $\operatorname{ran} J^* \supset \operatorname{ran} T$ in $\mathcal{H}_T$ therefore $\ker J^{**} = (\operatorname{ran} J^*)^{\perp} = \{0\}$ in $\mathcal{H}_T$. Consequently we have that

$$\ker T_N = \ker J^{**}J^* = \ker J^* = (\operatorname{ran} J)^{\perp} = (\operatorname{ran} T)^{\perp}.$$

To complete the proof of (v) we observe that $(\operatorname{ran} T)^{\perp} \subset \mathcal{D}_* [T]$. Turning to the proof of (vi) we recall $J^{**} = \overline{J}$ so that

$$\begin{aligned}
\operatorname{dom} T_N &= \operatorname{dom} J^{**} J^* = \{g \in \operatorname{dom} J^* : J^* g \in \operatorname{dom} J^{**}\} \\
&= \left\{g \in \mathcal{D}_* [T] : J^* g \in \operatorname{dom} \overline{J}\right\} \\
&= \{g \in \mathcal{D}_* [T] : \exists \{f_n\} \subset \operatorname{dom} T, \langle J^* g - T f_n, J^* g - T f_n\rangle \to 0, \\
&\quad (T (f_m - f_n), f_m - f_n) \to 0, (T (f_m - f_n), T (f_m - f_n)) \to 0\} \\
&= \{g \in \mathcal{D}_* [T] : \exists \{f_n\} \subset \operatorname{dom} T, \forall f \in \operatorname{dom} T \;\; \langle T f, J^* g - T f_n\rangle \to 0, \\
&\quad (T (f_m - f_n), f_m - f_n) \to 0, (T (f_m - f_n), T (f_m - f_n)) \to 0\} \\
&= \{g \in \mathcal{D}_* [T] : \exists \{f_n\} \subset \operatorname{dom} T, \forall f \in \operatorname{dom} T \;\; (T f, g) = \lim \langle T f, T f_n\rangle, \\
&\quad (T (f_m - f_n), f_m - f_n) \to 0, (T (f_m - f_n), T (f_m - f_n)) \to 0\} \\
&= \{g \in \mathcal{D}_* [T] : \exists \{f_n\} \subset \operatorname{dom} T, \forall f \in \operatorname{dom} T \\
&\quad (T f, g) = \lim (f, T f_n), (T (f_m - f_n), f_m - f_n) \to 0\} \\
&= \{g \in \mathcal{D}_* [T] : \exists h \in \mathcal{R} [T], (T f, g) = (f, h) \;\; (f \in \operatorname{dom} T)\}.
\end{aligned}$$

Finally we determine the range of T_N in (vii):

$$\begin{aligned}
\operatorname{ran} T_N &= \operatorname{ran} J^{**} J^* = \{J^{**} \eta : \eta \in \operatorname{ran} J^* \cap \operatorname{dom} J^{**}\} \\
&= \left\{\overline{J} \eta : \eta \in \operatorname{ran} J^* \cap \operatorname{dom} \overline{J}\right\} \\
&= \Big\{\lim_{\langle \cdot, \cdot \rangle} T f_n : \{f_n\} \subset \operatorname{dom} T, (T f_m - T f_n, f_m - f_n) \to 0, \\
&\quad (T f_m - T f_n, T f_m - T f_n) \to 0, \lim_{\langle \cdot, \cdot \rangle} T f_n \in \operatorname{ran} J^* \Big\} \\
&= \{\lim T f_n : \{f_n\} \subset \operatorname{dom} T, (T f_m - T f_n, f_m - f_n) \to 0, \\
&\quad (T f_m - T f_n, T f_m - T f_n) \to 0, \\
&\quad \sup \left\{\lim |\langle T f, T f_n\rangle|^2 : f \in \operatorname{dom} T, (T f, T f) \leq 1\right\} < \infty\Big\} \\
&= \{g \in \mathcal{H} : \exists \{f_n\} \subset \operatorname{dom} T, (T f_n - g, T f_n - g) \to 0, \\
&\quad (T f_m - T f_n, f_m - f_n) \to 0, \\
&\quad \sup \left\{|(f, g)|^2 : f \in \operatorname{dom} T, (T f, T f) \leq 1\right\} < \infty\Big\} \\
&= \mathcal{R} [T] \cap \mathcal{R}^* [T].
\end{aligned}$$

We are now in a position to determine T_N as follows. Take the orthogonal decomposition $\overline{\operatorname{ran} T} \bigoplus (\operatorname{ran} T)^{\perp} = \mathcal{H}$. T_N is identically 0 on $(\operatorname{ran} T)^{\perp}$, $\overline{\operatorname{ran} T_N} = \overline{\operatorname{ran} T}$ and T_N is one-to-one on $\operatorname{dom} T_N \cap \overline{\operatorname{ran} T}$. More precisely for every $h \in \operatorname{ran} T_N \subset \overline{\operatorname{ran} T}$ there exists a unique $g \in \operatorname{dom} T_N \cap \overline{\operatorname{ran} T}$ such that $T_N g = h$.

□

Theorem 3 *Let T be an operator on $\mathcal{H}$ such that its Krein-von Neumann extension exists. The following are equivalent:*
(a) *T_N is invertible.*
(b) $\operatorname{ran} T$ *is dense in $\mathcal{H}$.*
If this is the case then the Friedrichs extension of T^{-1} exists and satisfies the following identity:

$$T_N^{-1} = \left(T^{-1}\right)_F.$$

Proof. We know from above that $\ker T_N = (\operatorname{ran} T)^{\perp}$. Since (a) involves that $\ker T_N = \{0\}$, (a) and (b) are equivalent indeed. In this case $\operatorname{dom} T^{-1} = \operatorname{ran} T$ is dense in $\mathcal{H}$, so it follows that T^{-1}

has its Friedrichs extension $\left(T^{-1}\right)_F$. But if S is a positive self-adjoint extension of T then S is invertible since $\operatorname{ran} S \supset \operatorname{ran} T$ and thus $\operatorname{ran} S$ is dense in $\mathcal{H}$ therefore $\ker S = (\operatorname{ran} S)^{\perp} = \{0\}$. At the same time S^{-1} is a positive self-adjoint extension of T^{-1} consequently $S^{-1} \prec \left(T^{-1}\right)_F$ according to the maximality of the Friedrichs extension [8], [11]. But then $S \succ \left(T^{-1}\right)_F^{-1}$ holds for any positive self-adjoint extension S of T, i.e. $\left(T^{-1}\right)_F^{-1}$ is the Krein-von Neumann extension of T. So it follows that $T_N^{-1} = \left(T^{-1}\right)_F$.

□

Theorem 4 *Let T, S be operators on $\mathcal{H}$ such that S is positive self-adjoint. The next properties are equivalent:*
(i) $|(Th, g)|^2 \leq (Th, h)(Sg, g)$ $(h \in \operatorname{dom} T, g \in \operatorname{dom} S)$.
(ii) T has a positive self-adjoint extension smaller than S.

Proof. Assume (i). Then $\operatorname{dom} S \subset \mathcal{D}_*[T]$ therefore $\mathcal{D}_*[T]$ is dense in $\mathcal{H}$ as $\operatorname{dom} S$ is so and Theorem 1 applies. Therefore T_N exists, $\operatorname{dom} S \subset \operatorname{dom} T_N^{1/2}$ $(= \mathcal{D}_*[T])$ and

$$\left(T_N^{1/2} g, T_N^{1/2} g\right) \leq (Sg, g) = \left(S^{1/2} g, S^{1/2} g\right) \quad (g \in \operatorname{dom} S)$$

Therefore by [3], Lemma 2.1, $T_N \prec S$ follows. Conversely, from (ii) implies by Theorem 1 that the smallest positive self-adjoint extension, the Krein-von Neumann extension T_N of T exists and is smaller than S. So $\operatorname{dom} S \subset \operatorname{dom} S^{1/2} \subset \operatorname{dom} T_N^{1/2}$ and

$$\begin{aligned}\left(T_N^{1/2} g, T_N^{1/2} g\right) &= \sup\left\{|(Th, g)|^2 : h \in \operatorname{dom} T, (Th, h) \leq 1\right\} \leq (Sg, g) \\ &\quad \text{for all } g \in \operatorname{dom} S.\end{aligned}$$

Thus (i) holds.

□

Theorem 5 *Let S be a positive self-adjoint extension of the given operator T on the Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*
(i) $S = T_N$.
(ii) $\{g \in \mathcal{D}_*[T] : \exists h \in \mathcal{R}[T], (Tf, g) = (f, h) \quad (f \in \operatorname{dom} T)\} \subset \operatorname{dom} S^{1/2}$,
$\left(S^{1/2} f, S^{1/2} f\right) \leq (T_N f, f)$ $\forall f \in \operatorname{dom} T_N$.

Proof. The fact that the Krein-von Neumann extension T_N is the smallest among all positive self-adjoint extensions assures that $T_N \prec S$ and thus (i) is equivalent to the statement $S \prec T_N$. In view of [3], Lemma 2.1, (ii) is nothing else than the relation $S \prec T_N$, hence (i) is equivalent to (ii).

□

3 The Friedrichs extension

We present the classical Friedrichs theorem with a recent proof by V. Prokaj and Z. Sebestyén so that its characterization appropriately follows. The Krein-von Neumann extension is characterized essentially simpler in the densely defined case.

Theorem 6 *The following equivalent statements are valid of a positive linear operator T, given on a Hilbert space $\mathcal{H}$:*
(i) T admits the Friedrichs extension T_F.
(ii) T is densely defined.
(iii) $\mathcal{D}[T]$ is dense in $\mathcal{H}$.
If this is the case then

$$\left(T_N^{1/2} g, T_N^{1/2} g\right) = \left(T_F^{1/2} g, T_F^{1/2} g\right) \quad \left(g \in \operatorname{dom} T_F^{1/2}\right).$$

Proof. (i) implies (ii) immediately since any positive self-adjoint extension of T is given arbitrarily on $(\operatorname{dom} T)^{\perp}$ contradicting to the uniqueness of the maximal extension T_F. As $\operatorname{dom} T \subset \mathcal{D}[T]$ and $\mathcal{D}[T] \subset \overline{\operatorname{dom} T}$, $\overline{\operatorname{dom} T} = \overline{\mathcal{D}[T]}$ holds and so (ii) is equivalent to (iii). Assuming (ii), Theorem 1 applies since $\operatorname{dom} T \subset \mathcal{D}_*[T]$. More precisely we argue according to the proof of Theorem 1 by considering $Q : \mathcal{H} \to \mathcal{H}_T$, $Q \doteq J^* \mid_{\operatorname{dom} T}$. Then Q is a densely defined closable operator as a restriction of a closed one, namely J^*. Quoting once more von Neumann, Q^*Q^{**} is a positive self-adjoint extension of T as for each f from $\operatorname{dom} T$ we find that

$$Q^*Q^{**}f = Q^*(Qf) = Q^*(Tf) = J^{**}(Tf) = J(Tf) = Tf \in \mathcal{H},$$

where we use the fact that $Q^* \supset J^{**} \supset J$. It is easy to determine the domain of this extension as follows:

$$\begin{aligned}
\operatorname{dom} Q^*Q^{**} &= \{g \in \operatorname{dom} Q^{**} : Q^{**}g \in \operatorname{dom} Q^*\} \\
&= \{g \in \mathcal{D}[T] : \\
&\quad \sup\{|\langle Qf, Q^{**}g\rangle| : f \in \operatorname{dom} T, (f,f) \leq 1\} < \infty\} \\
&= \{g \in \mathcal{D}[T] : \\
&\quad \sup\{|\langle Tf, Q^{**}g\rangle| : f \in \operatorname{dom} T, (f,f) \leq 1\} < \infty\} \\
&= \{g \in \mathcal{D}[T] : \\
&\quad \sup\{|(Q^*(Tf), g)| : f \in \operatorname{dom} T, (f,f) \leq 1\} < \infty\} \\
&= \{g \in \mathcal{D}[T] : \\
&\quad \sup\{|(J(Tf), g)| : f \in \operatorname{dom} T, (f,f) \leq 1\} < \infty\} \\
&= \{g \in \mathcal{D}[T] : \sup\{|(Tf, g)| : f \in \operatorname{dom} T, (f,f) \leq 1\} < \infty\} \\
&= \{g \in \mathcal{D}[T] : g \in \operatorname{dom} T^*\} = \mathcal{D}[T] \cap \operatorname{dom} T^*,
\end{aligned}$$

using the fact that $\operatorname{dom} Q^{**} = \operatorname{dom} \overline{Q} = \mathcal{D}[T]$. The range characterization of Q^*Q^{**} seems to be easier:

$$\operatorname{ran} Q^*Q^{**} = \{T^*f : f \in \operatorname{dom} Q^*Q^{**}\} = \{T^*f : f \in \mathcal{D}[T] \cap \operatorname{dom} T^*\}$$

because any self-adjoint extension of T is a restriction of its (existing) T^*. Another characterization is as follows (see [13]):

$$\begin{aligned}
\operatorname{ran} Q^*Q^{**} &= \{g \in \mathcal{H} : \sup\{|(f,g)| : f \in \operatorname{dom} Q^*Q^{**}, (Q^*Q^{**}f, Q^*Q^{**}f) \leq 1\} < \infty\} \\
&= \{g \in \mathcal{H} : \sup\{|(f,g)| : f \in \mathcal{D}[T] \cap \operatorname{dom} T^*, (T^*f, T^*f) \leq 1\} < \infty\}.
\end{aligned}$$

We also know (see [10]) that

$$\begin{aligned}
\operatorname{dom}(Q^*Q^{**})^{1/2} &= \operatorname{dom} Q^{**} = \operatorname{dom} \overline{Q} = \mathcal{D}[T], \\
\operatorname{ran}(Q^*Q^{**})^{1/2} &= \operatorname{ran} Q^* \\
&= \{g \in \mathcal{H} : \sup\{|(f,g)| : f \in \operatorname{dom} Q, \langle Qf, Qf\rangle \leq 1\} < \infty\} \\
&= \{g \in \mathcal{H} : \sup\{|(f,g)| : f \in \operatorname{dom} T, (Tf,f) \leq 1\} < \infty\}.
\end{aligned}$$

Finally we need for any $g \in \operatorname{dom}(Q^*Q^{**})^{1/2} = \mathcal{D}[T]$ the quantity

$$\begin{aligned}
\left((Q^*Q^{**})^{1/2} g, (Q^*Q^{**})^{1/2} g\right) &= (Q^{**}g, Q^{**}g) \\
&= \sup\left\{|\langle Tf, Q^{**}g\rangle|^2 : f \in \operatorname{dom} T, (Tf,f) \leq 1\right\} \\
&= \sup\left\{|(Q^*(Tf), g)|^2 : f \in \operatorname{dom} T, (Tf,f) \leq 1\right\} \\
&= \sup\left\{|(Tf, g)|^2 : f \in \operatorname{dom} T, (Tf,f) \leq 1\right\} \\
&= \left((J^{**}J^*)^{1/2} g, (J^{**}J^*)^{1/2} g\right),
\end{aligned}$$

according to Theorem 1. Now, if S is an arbitrary positive self-adjoint extension of T then, applying the same argumentation for S instead of T above, we find that

$$\begin{aligned} \operatorname{dom} S^{1/2} &= \mathcal{D}[S] \supset \mathcal{D}[T] = \operatorname{dom}(Q^*Q^{**})^{1/2} \\ \operatorname{ran} S^{1/2} &= \{g \in \mathcal{H} : \sup\{|(f,g)| : f \in \operatorname{dom} S, (Sf,f) \leq 1\} < \infty\} \\ &\subset \{g \in \mathcal{H} : \sup\{|(f,g)| : f \in \operatorname{dom} T, (Tf,f) \leq 1\} < \infty\} \\ &= \operatorname{ran}(Q^*Q^{**})^{1/2}. \end{aligned}$$

$\left(S^{1/2}f, S^{1/2}f\right) = \lim (Sf_n, f_n) = \lim (Tf_n, f_n) = \left((Q^*Q^{**})^{1/2} f, (Q^*Q^{**})^{1/2} f\right)$ also holds true for every $f \in \mathcal{D}[T]$ by choosing an appropriate sequence $\{f_n\} \subset \operatorname{dom} T$ so that $(f - f_n, f - f_n) \to 0$ and $(Tf_m - Tf_n, f_m - f_n) \to 0$ where $Sf_n = Tf_n$. These last considerations show that Q^*Q^{**} is the largest positive self-adjoint extension, the Friedrichs extension of T.

□

Theorem 7 *Let T be a densely defined positive linear operator on the Hilbert space $\mathcal{H}$. Then its Krein-von Neumann extension T_N, the smallest among all positive self-adjoint extensions, exists and is the restriction of the adjoint T^* of T to the subspace* $\operatorname{dom} T_N = \{g \in \operatorname{dom} T^* : T^*g \in \mathcal{R}[T]\}$. *The range of T_N is just* $\operatorname{ran} T_N = \mathcal{R}[T] \cap \operatorname{ran} T^*$.

Proof. In Theorem 2 we have that

$$\operatorname{dom} T_N = \{g \in \mathcal{D}_*[T] : \exists h \in \mathcal{R}[T], (Tf,g) = (f,h) \quad (f \in \operatorname{dom} T)\}.$$

Here if T is densely defined $g \in \operatorname{dom} T_N$ implies $g \in \operatorname{dom} T^*$ and $h = T^*g$. On the other hand for $g \in \operatorname{dom} T^*, T^*g = h \in \mathcal{R}[T]$ there exists $\{f_n\} \subset \operatorname{dom} T$ such that $(h - Tf_n, h - Tf_n) \to 0$ and $(Tf_m - Tf_n, f_m - f_{,n}) \to 0$. Therefore, since

$$|(f, Tf_n)|^2 = |(Tf, f_n)|^2 \leq (Tf,f)\,(Tf_n, f_n)$$

one has

$$|(Tf,g)|^2 = |(f,h)|^2 = \lim |(f,Tf_n)|^2 = \lim |(Tf,f_n)|^2 \leq (Tf,f) \lim (Tf_n, f_n)$$

where $m \doteq \lim (Tf_n, f_n) < \infty$; so g belongs to $\mathcal{D}_*[T]$ at once. From Theorem 2 we also obtain that

$$\operatorname{ran} T_N = \mathcal{R}[T] \cap \mathcal{R}^*[T]$$

where in case T is densely defined, $\mathcal{R}^*[T]$ is just $\operatorname{ran} T^*$ (see [13]).

□

Now we extend Ando-Nishio's result [2], Corollary 4, by giving a complete proof for it.

Theorem 8 *Let T be a densely defined positive operator on the Hilbert space $\mathcal{H}$. The following properties are equivalent:*
(i) The Friedrichs extension T_F of T is invertible.
(ii) The operator T is invertible and its inverse T^{-1} admits the Krein-von Neumann extension $\left(T^{-1}\right)_N$.
(iii) If for a vector f in $\mathcal{H}$ there exists $\{f_n\} \subset \operatorname{dom} T$ such that

$$(f - f_n, f - f_n) \to 0 \text{ and } (Tf_n, f_n) \to 0$$

then $f = 0$.
Moreover, if this is the case then

$$T_F^{-1} = \left(T^{-1}\right)_N.$$

Proof. Theorem 2 and its proof implies that

$$\begin{aligned} \ker T_F &= \ker (Q^*Q^{**})^{1/2} = \ker Q^{**} = \ker \overline{Q} \\ &= \{f \in \mathcal{H} : \exists \{f_n\} \subset \operatorname{dom} Q, (f - f_n, f - f_n) \to 0, \langle Qf_n, Qf_n \rangle \to 0\} \\ &= \{f \in \mathcal{H} : \exists \{f_n\} \subset \operatorname{dom} T, (f - f_n, f - f_n) \to 0, (Tf_n, f_n) \to 0\} \end{aligned}$$

hence (i) and (iii) are equivalent indeed. Assume now (i) and prove (ii). Clearly T is invertible since its extension T_F is, in fact the positive self-adjoint operator T_F^{-1} extends T^{-1} so T^{-1} admits the Krein-von Neumann extension $\left(T^{-1}\right)_N$ due to Theorem 1. Conversely, observe that the existing $\left(T^{-1}\right)_N$ is automatically invertible since $\operatorname{ran}\left(T^{-1}\right)_N \supset \operatorname{ran} T^{-1} = \operatorname{dom} T$ and $\operatorname{dom} T$ is dense by assumption. Since $\left(T^{-1}\right)_N^{-1}$ is a positive self-adjoint extension of T, $\ker T_F \subset \ker \left(T^{-1}\right)_N^{-1} = \{0\}$ so T_F is invertible. Since for any positive self-adjoint extension S of T^{-1} holds that $\operatorname{ran} S \supset \operatorname{ran} T^{-1} = \operatorname{dom} T$ where $\operatorname{dom} T$ is dense by assumption, S is invertible and the positive self-adjoint operator S^{-1} extends T, therefore $S^{-1} \prec T_F$ that is $T_F^{-1} \prec S$. So $T_F^{-1} = \left(T^{-1}\right)_N$ in view of Theorem 1.

□

4 Extensions with bounded inverse

Theorem 9 *Let T be an operator on the Hilbert space $\mathcal{H}$ such that T has a positive self-adjoint extension. The following statements are equivalent:*
(i) $\operatorname{ran} T$ is dense in $\mathcal{H}$ (therefore T is invertible) and $\left(T^{-1}\right)_F$ is bounded.
(ii) T_N is invertible and T_N^{-1} is bounded.
(iii) $0 < \inf \{\sup \{|(Tf, g)| : f \in \operatorname{dom} T, (Tf, f) \leq 1\} : g \in \mathcal{D}_* [T], (g, g) = 1\}$.
(iv) $\operatorname{ran} T$ is dense in $\mathcal{H}$ and $\sup \{(f, f) : f \in \operatorname{dom} T, (Tf, f) \leq 1\} < \infty$.

Proof. In view of Theorem 3 statements (i) and (ii) are equivalent. On the other hand T_N^{-1} is bounded if and only if T_N is bounded from below that is $T_N^{1/2}$ is bounded from below. But for each $g \in \operatorname{dom} T_N^{1/2} = \mathcal{D}_* [T]$ we have by Theorem 1 that

$$\left(T_N^{1/2} g, T_N^{1/2} g\right) = \sup \left\{|(Tf, g)|^2 : f \in \operatorname{dom} T, (Tf, f) \leq 1\right\} \quad \left(g \in \operatorname{dom} T_N^{1/2}\right)$$

hence (ii) and (iii) are equivalent. Applying a similar reasoning for $\left(T^{-1}\right)_F$, for each $h \in \operatorname{dom}\left(T^{-1}\right)_F$ we find that (see the proof of Theorem 6)

$$\begin{aligned} \left(\left(\left(T^{-1}\right)_F\right)^{1/2} h, \left(\left(T^{-1}\right)_F\right)^{1/2} h\right) &= \sup \left\{\left|\left(T^{-1} g, h\right)\right|^2 : g \in \operatorname{dom} T^{-1}, \left(T^{-1} g, g\right) \leq 1\right\} \\ &= \sup \left\{|(f, h)|^2 : f \in \operatorname{dom} T, (f, Tf) \leq 1\right\}. \end{aligned}$$

Hence $\left(T^{-1}\right)_F$ equivalently $\left(T^{-1}\right)_F^{1/2}$ is bounded if and only if

$$\sup \left\{|(f, h)|^2 : f \in \operatorname{dom} T, (f, Tf) \leq 1, h \in \mathcal{D}\left[T^{-1}\right], (h, h) \leq 1\right\} < \infty.$$

So (i) implies property (iv) because $\mathcal{D}\left[T^{-1}\right]$ $(\supset \operatorname{ran} T)$ is dense in $\mathcal{H}$. A similar reasoning shows that (i) follows from (iv).

□

Theorem 10 *Let T be a densely defined positive linear operator. Each of the following statements implies the other two:*
(i) T_F is invertible and T_F^{-1} is bounded.
(ii) T is invertible, $\left(T^{-1}\right)_N$ exists and is bounded.
(iii) $\{f \in \mathcal{H} : f \in \operatorname{dom} T, (f, Tf) \leq 1\}$ is bounded in $\mathcal{H}$.

Proof. That $(i) \Leftrightarrow (ii)$ follows by Theorem 8. Assume (iii) and conclude at once that T is invertible since

$$\ker T \subset \{f \in \mathcal{H} : f \in \operatorname{dom} T, (f, Tf) \leq 1\}.$$

It follows that

$$m \doteq \sup\left\{\left(T^{-1}g, T^{-1}g\right) : g \in \operatorname{ran} T, \left(T^{-1}g, g\right) \leq 1\right\} < \infty.$$

Therefore $\mathcal{D}_*\left[T^{-1}\right]$ equals $\mathcal{H}$ as for each f in $\mathcal{H}$ we have:

$$\left|\left(T^{-1}g, f\right)\right|^2 \leq \left(T^{-1}g, T^{-1}g\right) \cdot (f, f) \leq m \cdot (f, f) \cdot \left(T^{-1}g, g\right) \quad \left(g \in \operatorname{dom} T^{-1}\right).$$

This implies that $\left(T^{-1}\right)_N$ exists and $\left(T^{-1}\right)_N^{1/2}$ is an everywhere defined self-adjoint operator, that is bounded, so $\left(T^{-1}\right)_N$ is bounded as well. Since $\left(T^{-1}\right)_N = T_F^{-1}$ (T_F^{-1} exists, see Theorem 8), T_F^{-1} is bounded and (iii) implies (i) and (ii). Finally assuming (ii), T^{-1} necessarily fulfils the so called Schwartz-inequality:

$$\left(T^{-1}g, T^{-1}g\right) \leq m \cdot \left(T^{-1}g, g\right) \quad \left(g \in \operatorname{dom} T^{-1}\right)$$

that follows (iii).

□

5 Extensions with compact inverse

Theorem 11 *Let T be an operator on the Hilbert space $\mathcal{H}$ such that T has a positive self-adjoint extension. The following statements are equivalent:*
(i) $\operatorname{ran} T$ *is dense in $\mathcal{H}$ (therefore T is invertible) and* $\left(T^{-1}\right)_F$ *is compact.*
(ii) T_N *is invertible and* $(T_N)^{-1}$ *is compact.*
(iii) $\operatorname{ran} T$ *is dense in $\mathcal{H}$ and* $\forall \varepsilon > 0\ \exists f_1, ..., f_n \in \operatorname{dom} T, (Tf_m, Tf_m) \leq 1\ (m = 1, 2, ..., n)$*:* $\forall f \in \operatorname{dom} T, (Tf, Tf) \leq 1\ \exists m \in \{1, ..., n\} : (T(f - f_m), f - f_m) < \varepsilon$.

Proof. The first two statements are equivalent as Theorem 8 applies and $T_N^{-1} = \left(T^{-1}\right)_F$ here. But $\left(T^{-1}\right)_F$ is compact if and only if the corresponding Q in its factorization (see Theorem 6) is compact. Even this fact is expressed in (iii) so (i) and (iii) are equivalent.

□

Theorem 12 *Let T be a densely defined positive linear operator on the Hilbert space $\mathcal{H}$. The following statements are equivalent:*
(i) T_F *is invertible and* T_F^{-1} *is compact.*
(ii) T *is invertible,* $\left(T^{-1}\right)_N$ *exists and is compact.*
(iii) $\{f \in \mathcal{H} : f \in \operatorname{dom} T, (f, Tf) \leq 1\}$ *is precompact in $\mathcal{H}$.*

Proof. As Theorem 8 applies both in (i) and (ii), and $\left(T^{-1}\right)_N = T_F^{-1}$, these properties are clearly equivalent. Now, $\left(T^{-1}\right)_N$ exists and is compact if and only if the corresponding J in its factorization (see Theorem 1) is compact, as stated in (iii). Of course, T_F^{-1} exists and is compact in the only case when the corresponding Q factor (see the proof of Theorem 6) is compactly invertible as expressed in (iii).

□

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